SETS OF UNIQUENESS FOR DIRICHLET-TYPE SPACES

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ABSTRACT. We study the uniqueness sets on the unit circle for weighted Dirichlet spaces.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc in the complex plane, and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. Let H^2 denote the Hardy space of analytic functions on $\mathbb{D}$. If μ is a positive Borel measure on the unit circle $\mathbb{T}$, the Dirichlet-type space $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P\mu(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$ and $P\mu$ is the Poisson integral of μ

$$P\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Since $\mathcal{D}(\mu) \subset H^2$, every function $f \in \mathcal{D}(\mu)$ has non-tangential limits almost everywhere on $\mathbb{T}$. We denote by $f(\zeta)$ the non-tangentiel limit of f at $\zeta \in \mathbb{T}$ if it exists. It turns out that there is a useful formula for expressing the norm of the Dirichlet-type space in terms of the local Dirichlet integral

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty.$$

where $\mathcal{D}_\xi(f)$ is the local Dirichlet integral of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\xi)|^2}{|e^{it} - \xi|^2} \frac{dt}{2\pi}.$$

For a proof of this see [13, Proposition 2.2]. Note that if $d\mu(e^{it}) = dt/2\pi$, the normalized arc measure on $\mathbb{T}$, then the space $\mathcal{D}(\mu)$ coincides with the classical space of functions with finite Dirichlet integral. These spaces were introduced by Richter [11] and generalized by Aleman [1] for nonnegative finite Borel measure on $\overline{\mathbb{D}}$. The spaces $\mathcal{D}(\mu)$ were studied in [1, 11, 12, 13, 14, 15, 17].

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Let $\mathcal{D}^h(\mu)$ be the harmonic version of $\mathcal{D}(\mu)$ given by

$$\mathcal{D}^h(\mu) := \left\{ f \in L^2(\mathbb{T}) \,:\, D_\mu(f) < \infty \right\}.$$

We define the capacity C_μ of a set $E \subset \mathbb{T}$ by

$$C_\mu(E) := \inf \left\{ \|f\|_\mu^2 \,:\, f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ a.e. on a neighborhood of } E \right\},$$

see [4, 5]. If $C_\mu(E) = 0$, then E has Lebesgue measure zero. Indeed, if $C_\mu(E) = 0$, then there exists a sequence $(f_i) \in \mathcal{D}^h(\mu)$ such that $\|f_i\|_\mu \leq 2^{-i}$ and $|f_i| \geq 1$ a.e. on a neighborhood of E. Then $f = \sum_i f_i \in \mathcal{D}^h(\mu)$ and $|f| = \infty$ on E. We have $\infty > \mathcal{D}_\mu(f) \geq \int_E D_\xi(f) d\mu(\xi)$ and this forces E to have measure zero. We say that a property holds C_μ-quasi-everywhere (C_μ-q.e.) if it holds everywhere outside a set of zero C_μ capacity. Note that C_μ-q.e implies a.e. We have

$$C_\mu(E) := \inf \left\{ \|f\|_\mu^2 \,:\, f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \ C_\mu\text{-q.e on } E \right\}.$$

see [6, Theorem 4.2]. Every function $f \in \mathcal{D}(\mu)$ has non-tangential limits C_μ-quasi–everywhere on $\mathbb{T}$ [4, Theorem 2.1.9]. Let E be a subset of $\mathbb{T}$. The set E is said to be a uniqueness set for $\mathcal{D}(\mu)$ if, for each $f \in \mathcal{D}(\mu)$ such that it non-tangentiel limit $f = 0$ on E, we have $f = 0$.

In order to state our main result, we define some notions. Given $E \subset \mathbb{T}$, we write $|E|$ for the Lebesgue measure of E. For $w \in L^1(\mathbb{T})$, we denote by $I(w)$ the mean of w over I

$$I[w] = \frac{1}{|I|} \int_I w(\zeta)|d\zeta|.$$

A nonnegative function w is a Muckenhoupt A_2-weight if for all arc $I \subset \mathbb{T}$

$$\sup_{I \subset \mathbb{T}} I[w] I[w^{-1}] < +\infty.$$

Theorem 1.1. *Let μ be an absolutely continuous measure with respect to the Lebesgue measure on $\mathbb{T}$, $d\mu(\zeta) = w(\zeta)|d\zeta|$ and w is a Muckenhoupt A_2-weight. Let E be a Borel subset of $\mathbb{T}$ of Lebesgue measure zero. We assume that there exists a family of pairwise disjoint open arcs (I_n) of $\mathbb{T}$ such that $E \subset \bigcup_n I_n$. Suppose*

$$\sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} = -\infty;$$

then E is a uniqueness set for $\mathcal{D}(\mu)$.

The case of the Dirichlet space, $d\mu(\zeta) = |d\zeta|/2\pi$, was obtained by Khavin and Maz'ya [9]; see also [2, 3, 8]. In [10], we give the generalization of their result in the Dirichlet spaces $\mathcal{D}_s$, $0 < s \leq 1$, which consist of all analytic functions $f \in H^2$ such that

$$\mathcal{D}_s(f) := \int_\mathbb{T} \int_\mathbb{T} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^{1+s}} \frac{|d\zeta|}{2\pi} \frac{|d\xi|}{2\pi} \asymp \int_\mathbb{D} |f'(z)|^2 (1 - |z|^2)^{1-s} dA(z).$$

The remaining of the note is devoted to proof of the theorem.

2. Proof

To prove our theorem, we use the following lemmas,

Lemma 2.1. *Let w be a Muckenhoupt A_2-weight and let $d\mu(\zeta) = w(\zeta)|d\zeta|$, then*

(a) If I is an arc of $\mathbb{T}$ and ξ_I its center, then

$$|I| \int_{\mathbb{T}\setminus I} \frac{d\mu(\zeta)}{|\zeta - \xi_I|^2} \leq cI[w],$$

for some positive constant C independent of I.

(b) for all nonnegative function g and all arcs I of $\mathbb{T}$

$$\Big(\frac{1}{|I|}\int_I g(\zeta)|d\zeta|\Big)^2 \leq \frac{1}{\mu(I)}\int_I g(\zeta)^2 d\mu(\zeta).$$

(c) for all open arcs I,

$$\mu(I) \geq \frac{\mu(\mathbb{T})}{\pi c}|I|^2\Big(\log\frac{2\pi}{|I|}\Big)^2.$$

Proof. For a proof of (a) , see [7, Lemma 1] and [16] p.200 for (b). Let now to prove (c). By (b), we have $|\mathbb{T}\setminus I|^2/|\mathbb{T}|^2 \leq \mu(\mathbb{T}\setminus I)/\mu(\mathbb{T})$. By (a) and (b) we get

$$\frac{\mu(I)}{|I|^2} \geq \frac{1}{c}\int_{\mathbb{T}\setminus I}\frac{d\mu(\zeta)}{|\zeta-\xi_I|^2} \geq \frac{\mu(\mathbb{T}\backslash I)}{c}\Big(\frac{1}{|\mathbb{T}\backslash I|}\int_{\mathbb{T}\setminus I}\frac{|d\xi|}{|\zeta-\xi_I|}\Big)^2 \geq \frac{4}{c}\frac{\mu(\mathbb{T})}{|\mathbb{T}|^2}\Big(\log\frac{2\pi}{|I|}\Big)^2.$$

$\square$

Let I be an open arc of $\mathbb{T}$ and f be a function. We set

$$\mathcal{D}_{I,\mu}(f) := \int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^2}\frac{|dz|}{2\pi}d\mu(w) \qquad \text{and} \qquad m_I(f) := \frac{1}{|I|}\int_I |f(\xi)||d\xi|.$$

Lemma 2.2. *Let $d\mu = wdm$ be a measure such that $w \in (A_2)$. Suppose that $0<\gamma<1$. Let $E \subset \mathbb{T}$ and $f \in \mathcal{D}(\mu)$ be such that $f|E = 0$. Then, for any open arc $I \subset \mathbb{T}$ with $|I| \leq \gamma\pi$*

$$m_I(f)^2 \leq \kappa\frac{\mathcal{D}_{I,\mu}(f)}{C_\mu(E\cap I)},$$

where κ depending only on γ.

Proof. Without loss generality, we assume that $I = (e^{-i\theta}, e^{i\theta})$ with $\theta < \gamma\pi/2$. Let $J = (e^{-2i\theta/(1+\gamma)}, e^{2i\theta/(1+\gamma)})$ and $\widetilde{f}$ be such that

$$\widetilde{f}(e^{it}) = \begin{cases} f(e^{it}), & e^{it} \in I, \\ f(e^{i\frac{3\theta-|t|}{2}}), & e^{it} \in J\backslash I. \end{cases}$$

Then by a change of variable, we get

$$\mathcal{D}_{I,\mu}(f) \asymp \mathcal{D}_{J,\mu}(\widetilde{f}) \quad \textit{and} \quad m_I(f) \asymp m_J(\widetilde{f}), \tag{1}$$

where the implied constants depend only on γ, see [10].

Let $I_\gamma = (e^{-i\theta_\gamma}, e^{i\theta_\gamma})$ with $\theta_\gamma = \frac{3+\gamma}{2(1+\gamma)}\theta$. Note that $I \subset I_\gamma \subset J$. Let ϕ be a positive function on $\mathbb{T}$, $0 \leq \phi \leq 1$, such that $\operatorname{supp}\phi = I_\gamma$, $\phi = 1$ on I and

$$|\phi(z) - \phi(w)| \leq \frac{c_1}{|J|}|z - w|, \qquad z, w \in \mathbb{T}.$$

where c_1 depending only on γ.

Now, we consider the function

$$F(z) = \phi(z)\Big|1 - \frac{|\widetilde{f}(z)|}{m_J(\widetilde{f})}\Big|, \qquad z \in \mathbb{T}.$$

Hence $F \geq 0$ and $F = 1$ C_μ-q.e on $E \cap I$. Therefore,

$$C_\mu(E \cap I) \leq \|F\|_\mu^2. \tag{2}$$

We claim that

$$\|F\|_\mu^2 \leq \kappa \frac{\mathcal{D}_{I,\mu}(f)}{m_I(f)^2} \tag{3}$$

where κ depending only on γ. The Lemma 2.2 follows from (2) and (3).

Now, we prove the claim (3). We have

$$\begin{aligned}
\|F\|_\mu^2 &= \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi}\frac{d\mu(\xi)}{2\pi} \\
&\leq \frac{1}{m_J(\widetilde{f})^2}\int_J |m_J(\widetilde{f}) - |\widetilde{f}(\zeta)||^2 \frac{|d\zeta|}{2\pi} + \int_J\int_J \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}\frac{d\mu(\xi)}{2\pi} \\
&\qquad + \frac{1}{m_J(\widetilde{f})^2} \int_{\zeta \in \mathbb{T}\setminus J}\int_{\xi \in I_\gamma} \frac{|m_J(\widetilde{f}) - |\widetilde{f}(\xi)||^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}\frac{d\mu(\xi)}{2\pi} \\
&\qquad + \frac{1}{m_J(\widetilde{f})^2} \int_{\xi \in \mathbb{T}\setminus J}\int_{\zeta \in I_\gamma} \frac{|m_J(\widetilde{f}) - |\widetilde{f}(\zeta)||^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}\frac{d\mu(\xi)}{2\pi} \\
&\qquad = \frac{A}{2\pi m_J(\widetilde{f})^2} + \frac{B}{4\pi^2} + \frac{C}{4\pi^2 m_J(\widetilde{f})^2} + \frac{D}{4\pi^2 m_J(\widetilde{f})^2}.
\end{aligned} \tag{4}$$

Note that, by Lemma 2.1 (b)

$$|m_J(\widetilde{f}) - |\widetilde{f}(\zeta)||^2 \leq \Big(\frac{1}{|J|}\int_J |\widetilde{f}(\xi) - \widetilde{f}(\zeta)||d\xi|\Big)^2 \leq \frac{1}{\mu(J)}\int_J |\widetilde{f}(\xi) - \widetilde{f}(\zeta)|^2 d\mu(\xi).$$

Hence by (1) and Lemma 2.1 (c)

$$\begin{aligned} A &:= \int_J |m_J(\widetilde{f}) - |\widetilde{f}(\zeta)||^2 |d\zeta| \\ &\leq \frac{1}{\mu(J)} \int_J \int_J |\widetilde{f}(\xi) - \widetilde{f}(\zeta)|^2 d\mu(\xi)|d\zeta| \\ &\leq \frac{|J|^2}{\mu(J)} \int_J \int_J \frac{|\widetilde{f}(\xi) - \widetilde{f}(\zeta)|^2}{|\xi - \zeta|^2} d\mu(\xi)|d\zeta| \\ &\leq c_2 \mathcal{D}_{I,\mu}(f), \end{aligned} \tag{5}$$

where c_2 depending only on γ.

Let us now estimate B. If $(\zeta, \xi) \in J \times J$, then we write

$$\begin{aligned} |F(\zeta) - F(\xi)| &= \Big|\phi(\zeta)\Big(\Big|1 - \frac{|\widetilde{f}(\zeta)|}{m_J(\widetilde{f})}\Big| - \Big|1 - \frac{|\widetilde{f}(\xi)|}{m_J(\widetilde{f})}\Big|\Big) + (\phi(\zeta) - \phi(\xi))\Big|1 - \frac{|\widetilde{f}(\xi)|}{m_J(\widetilde{f})}\Big|\Big| \\ &\leq \frac{1}{m_J(\widetilde{f})}|\widetilde{f}(\zeta) - \widetilde{f}(\xi)| + \frac{c_1}{m_J(\widetilde{f})} \frac{|\zeta - \xi|}{|J|} |m_J(\widetilde{f}) - |\widetilde{f}(\xi)||. \end{aligned} \tag{6}$$

Note that, by Cauchy-Schwarz,

$$|m_J(\widetilde{f}) - \widetilde{f}(\xi)|^2 \leq \frac{1}{|J|} \int_J |\widetilde{f}(\eta) - \widetilde{f}(\xi)|^2 |d\eta|. \tag{7}$$

So, by (6), (7) and (1)

$$\begin{aligned} B &:= \int_J \int_J \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\ &\leq \frac{2}{m_J(\widetilde{f})^2} \int_J \int_J \frac{|\widetilde{f}(\zeta) - \widetilde{f}(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\ &\qquad + \frac{2c_1^2}{m_J(\widetilde{f})^2 |J|^3} \int_J \int_J \int_J |\widetilde{f}(\eta) - \widetilde{f}(\xi)|^2 |d\eta||d\zeta| d\mu(\xi) \\ &\leq \frac{2 + 2c_1^2}{m_J(\widetilde{f})^2} \int_J \int_J \frac{|\widetilde{f}(\eta) - \widetilde{f}(\xi)|^2}{|\eta - \xi|^2} |d\eta| d\mu(\xi) \\ &\leq c_3 \frac{\mathcal{D}_{I,\mu}(f)}{m_I(f)^2}. \end{aligned} \tag{8}$$

where c_3 depending only on γ.

Next, using again (7) and (1)

$$
\begin{aligned}
C &:= \int_{\zeta\in\mathbb{T}\setminus J}\int_{\xi\in I_\gamma}\frac{|m_J(\widetilde{f})-|\widetilde{f}(\xi)||^2}{|\zeta-\xi|^2}|d\zeta|d\mu(\xi)\\
&\leq \int_{\zeta\in\mathbb{T}\setminus J}\frac{|d\zeta|}{d(\zeta,I_\gamma)^2}\int_{\xi\in I_\gamma}|m_J(\widetilde{f})-|\widetilde{f}(\xi)||^2d\mu(\xi)\\
&\leq \frac{c_4}{|J|^2}\int_J\int_J|\widetilde{f}(\eta)-\widetilde{f}(\xi)|^2|d\eta|d\mu(\xi)\\
&\leq c_4\int_J\int_J\frac{|\widetilde{f}(\eta)-\widetilde{f}(\xi)|^2}{|\eta-\xi|^2}|d\eta|d\mu(\xi)\\
&\leq c_5\mathcal{D}_{I,\mu}(f), \qquad (9)
\end{aligned}
$$

where c_4, c_5 depend only on γ.

Finally, by Lemma (2.1) (a) and (b) and (1)

$$
\begin{aligned}
D &:= \int_{\xi\in\mathbb{T}\setminus J}\int_{\zeta\in I_\gamma}\frac{|m_J(\widetilde{f})-|\widetilde{f}(\zeta)||^2}{|\zeta-\xi|^2}|d\zeta|d\mu(\xi)\\
&\leq \int_{\xi\in\mathbb{T}\setminus J}\frac{d\mu(\xi)}{d(\xi,I_\gamma)^2}\int_{\zeta\in I_\gamma}|m_J(\widetilde{f})-|\widetilde{f}(\zeta)||^2|d\zeta|\\
&\leq c_6\frac{\mu(J)}{|J|^2}\int_{\zeta\in I_\gamma}\frac{1}{\mu(J)}\int_J|\widetilde{f}(\eta)-\widetilde{f}(\zeta)|^2d\mu(\eta)|d\zeta|\\
&\leq \frac{c_6}{|J|^2}\int_{I_\gamma}\int_J|\widetilde{f}(\eta)-\widetilde{f}(\zeta)|^2|d\eta|d\mu(\zeta)\\
&\leq c_6\int_J\int_J\frac{|\widetilde{f}(\eta)-\widetilde{f}(\zeta)|^2}{|\eta-\zeta|^2}|d\eta|d\mu(\zeta)\\
&\leq c_7\mathcal{D}_{I,\mu}(f), \qquad (10)
\end{aligned}
$$

where c_6, c_7 depend only on γ.

By (5), (8), (9) and (10) we get (3) and the proof is complete. □

Proof of Theorem 1.1. Since $|E| = 0$, we can assume that $\sup_n |I_n| \leq \gamma\pi$ with $\gamma \in (0,1)$. Let $f \in \mathcal{D}(\mu)$ be such that $f|E = 0$. We set $\ell = \sum_n |I_n|$. By Lemma 2.2 and Jensen's inequality

$$\begin{aligned}
\int_{\bigcup I_n} \log|f(\xi)||d\xi| &= \sum_n |I_n| \frac{1}{|I_n|}\int_{I_n} \log|f(\xi)||d\xi| \\
&\le \sum_n |I_n| \log \frac{1}{|I_n|}\int_{I_n} |f(\xi)||d\xi| \\
&\le \sum_n |I_n| \log\Big(\frac{\kappa \mathcal{D}_{I_n,\mu}(f)}{C_\mu(E\cap I_n)}\Big) \\
&= \sum_n |I_n| \log\frac{|I_n|}{C_\mu(E\cap I_n)} + \ell \sum_n \frac{|I_n|}{\ell}\log\Big(\frac{\kappa\mathcal{D}_{I_n,\mu}(f)}{|I_n|}\Big) \\
&\le \sum_n |I_n| \log\frac{|I_n|}{C_\mu(E\cap I_n)} + \ell\log\Big(\frac{\kappa}{\ell}\sum_n \mathcal{D}_{I_n,\mu}(f)\Big) \\
&\le \sum_n |I_n| \log\frac{|I_n|}{C_\mu(E\cap I_n)} + \ell\log\Big(\frac{\kappa}{\ell}D_\mu(f)\Big) = -\infty.
\end{aligned}$$

By Fatou's Theorem we obtain $f = 0$ and the proof is complete.

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